Trace Anomaly in Relativistic Stars

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The Planck scale

Classical gravity with matter,

$$T_{\mu\nu} = (8\pi G)^{-1} \left(R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} \right)$$

Now introduce quantum corrections for matter

$$\langle : \hat{T}_{\mu\nu} : \rangle \sim \hbar R^2 g_{\mu\nu} + \dots$$

Comparing components,

$$T_{\mu\nu} \approx \langle : \hat{T}_{\mu\nu} : \rangle \quad \Rightarrow \qquad R^{-1} \sim G\hbar \quad \leftrightarrow \quad \rho \sim \rho_P = M_P^4$$

We can neglect quantum effects of curvature below Planck scale.

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Outline

1) When does QFTCS become macroscopic?

2) Application: compact stars

Derivation:

Trace Anomaly

QFT in curved spacetime becomes relevant much before the Planck scale

Trace anomaly

QFT curved space:

$$T_{\mu\nu} = T^{(m)}_{\mu\nu} + T^{(A)}_{\mu\nu}$$

↙ ↘

'classical'

break conformal
symmetry classically

no explicit dependence $g_{\mu\nu}$

'quantum'

anomaly

explicitly dependent $g_{\mu\nu}$

Weyl anomaly: $T = T^{(m)} + T^{(A)}$

$$T^{(A)} = cW^2 - aE \qquad c, a > 0 \qquad \int E = \chi$$

Assumptions

Much below the Planck density $\rho_P = M_p^4$:

(A1) Semi-classical approximation is valid

$$G_{\mu\nu} = 8\pi G \left(T_{\mu\nu}^{(m)} + T_{\mu\nu}^{(A)} \right)$$

(A2) Matter equation of state is conformal at leading order

$$T^{(m)} = -\rho + 3p \underset{m^4 \ll \rho \ll \rho_P}{\approx} -m^{4(1-\alpha)} \rho^{\alpha} \qquad \alpha < 1$$

(A3) Individual components

$$|T_{\mu\nu}^{(A)}| \ll |T_{\mu\nu}^{(m)}|$$

Evaluation

Restrict to regions where $W^2 \approx 0$

Ricci decomposition $E = W^2 + 2\left(\frac{R^2}{3} - R_{\mu\nu}R^{\mu\nu}\right)$

Evaluate on background:

$$R = -GT^{(m)} = -Gm^{4(1-\alpha)}\rho^\alpha$$

$$\begin{aligned} R_{\mu\nu}R^{\mu\nu} &= G^2 T^{(m)\mu\nu}T^{(m)}_{\mu\nu} + \ldots + R^2 \\ &= G^2(\rho^2 + 3p^2) + \ldots + G^2\rho^{2\alpha} \approx G^2\rho^2 \end{aligned}$$

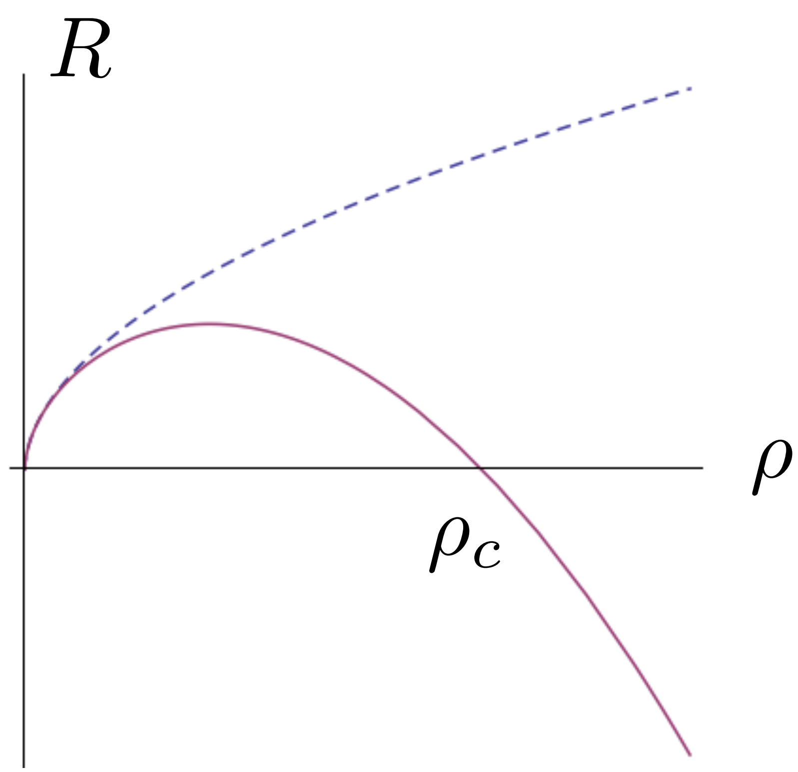
Punchline $T^{(A)} = -aE \sim G^2\rho^2$

Anomalous field equations

Include backreaction on trace of semi-classical equations:

$$R = -8\pi G \left(T^{(m)} + T^{(A)} \right)$$

Replacing all of this we find:



$$R \approx M_p^{-2} \left(m^{4(1-\alpha)} \rho^{\alpha} - M_p^{-4} \rho^2 \right)$$

$$\rho_c \sim (m/M_p)^{\frac{4(1-\alpha)}{2-\alpha}} \rho_p \ll \rho_p$$

e.g. for electrons $\rho_c \sim 10^{-30} \rho_p$

Summary (so far)

If you ask: when does $|T^{(A)}_{\mu\nu}| \sim |T^{(m)}_{\mu\nu}|$

the answer is* $\rho \sim \rho_p$

If you ask: when does $|T^{(A)}| \sim |T^{(m)}|$

the answer is $\rho_c \sim (m/M_p)^{\frac{4(1-\alpha)}{2-\alpha}} \rho_p \ll \rho_p$

Physical implications?

Application:

Compact stars

A regime where the trace anomaly becomes interesting is the Buchdahl limit

Buchdahl's Theorem

[Buchdahl '59]

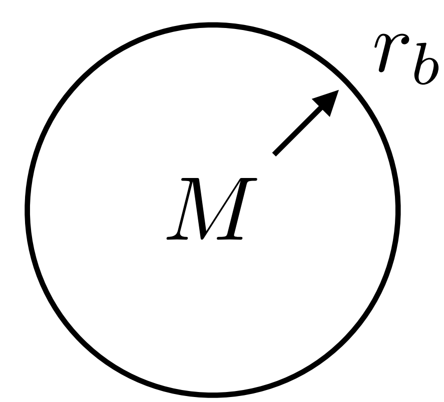
Static, spherically symmetric isotropic perfect fluid

$$T^{\mu}_{\nu} = \mathrm{diag}(-\rho, p, p, p)$$

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2 d\Omega^2$$

Then, assuming nothing about the EOS but only

$$\rho > 0 \quad , \quad \partial_r \rho \leq 0 \quad , \quad G_{\mu\nu} = 8\pi G T_{\mu\nu}$$



$$f(0) \geq 0 \quad \Rightarrow \quad \frac{r_b}{GM} \geq \frac{9}{4}$$

local

global

Tortoise coordinate

$$\frac{dr_*}{dr} = \sqrt{h(r)/f(r)}$$

Exterior $\quad r_* = r + 2M \log\left(r/2M - 1\right)$

Black hole: $\quad r_*(r \to 2M) \to -\infty$

Buchdahl limit

Regular star: $\quad \dfrac{dr_*}{dr} = \dfrac{1}{\sqrt{f(0)}} + \mathcal{O}(r)$ $\qquad f(0) \to 0$

Example: cold fermions

Density of states

$$\frac{d\mathcal{N}}{d^3x\, d^3p} = \frac{g}{h^3} f(x,p) \qquad\qquad f = \left(e^{\beta(E-\mu)} + 1\right)^{-1}$$

Low temperature

- Non-relativistic: $p \sim \rho^{5/3}$
- Intermediate: misleadingly called 'ultra-relativistic' $p \sim \rho^{4/3}$
- High energy $\rho - 3p = m^2\sqrt{\rho}$

At leading order it behaves like a CFT

Since quantum is identically conserved $\nabla_\mu T^{(A)\mu\nu} = 0$

$$\nabla_\mu T^{(m)\mu\nu} = 0 \qquad \frac{f'}{f} = -2\frac{p'}{p+\rho}$$

Integrate inside the sphere using Fermi-Dirac

$$\log\frac{f(r_b)}{f(0)} = -2\int_{p(0)\to\infty}^{p(r_b)=0}\frac{dp}{p+\rho} \sim \frac{1}{2}\log\frac{p(0)}{m^4}$$

$$\frac{f(0)}{f(r_b)} \sim \frac{m^2}{\sqrt{\rho(0)}}$$

Buchdahl limit (background)

Buchdahl limit and anomaly

For the static, spherically symmetric isotropic perfect fluid

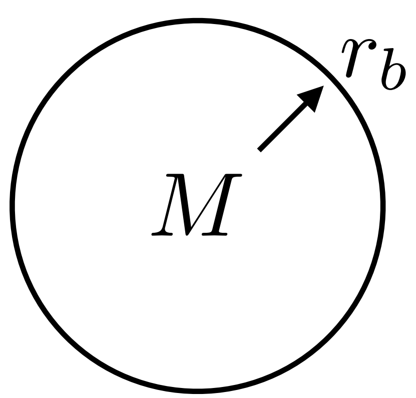
$$T^{(m)\mu}{}_{\nu} = \mathrm{diag}(-\rho, p, p, p)$$

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2 d\Omega^2$$

Regularity at the center requires

$$h(0) = 1 \quad , \quad f'(0) = 0 = h'(0)$$

$$\Rightarrow \quad W^2(0) = 0 \qquad\qquad W_{\mu\nu\rho\sigma}(0) \neq 0$$



The trace anomaly results are valid at the center

Wave equation

Consider a probe field propagating in this curved background

Massless minimally coupled scalar: $\Box\Phi = 0$

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2 d\Omega_2^2$$

$$-\partial_{r_*}^2 u + V(r_*)u = \omega^2 u \qquad \Phi = \frac{u(r)}{r} Y_m^\ell(\theta, \varphi) e^{-i\omega t}$$

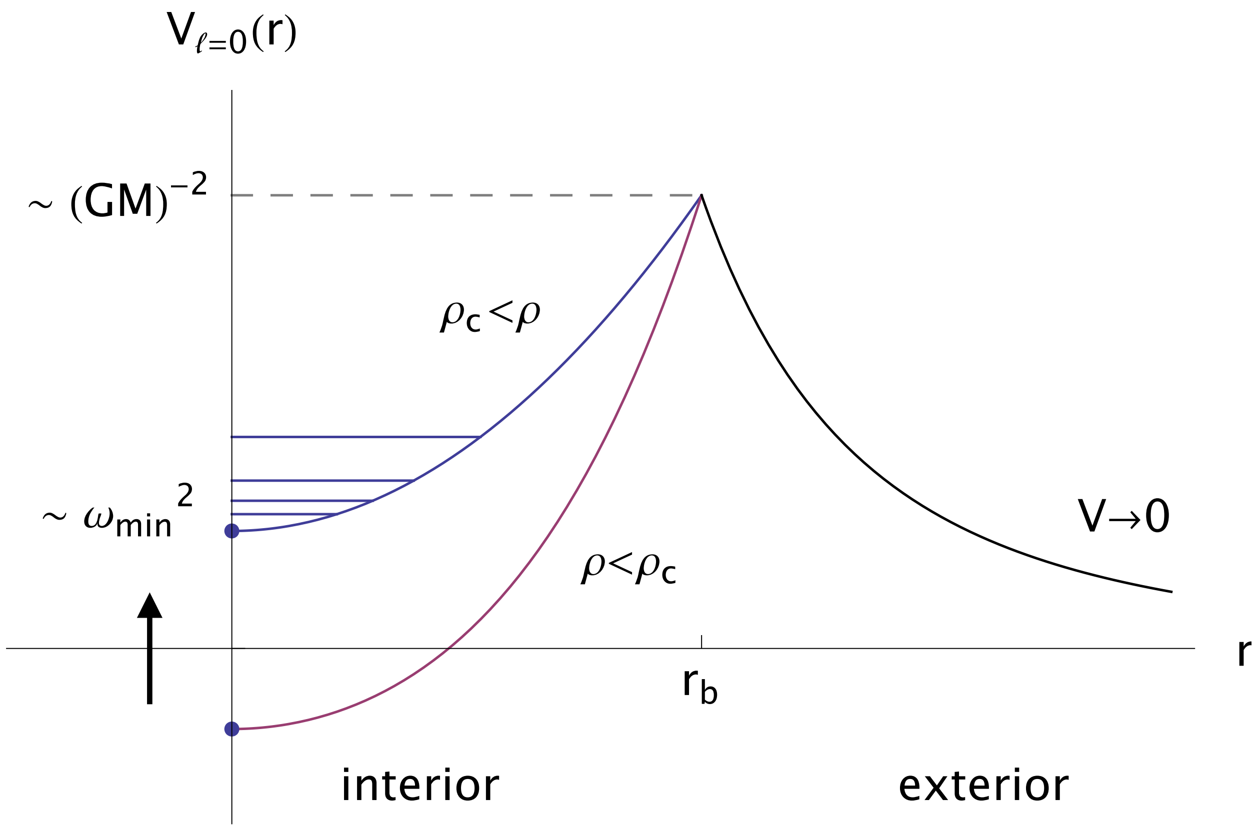
$$V_\ell = \frac{f}{2rh}\left(\frac{f'}{f} - \frac{h'}{h}\right) + \frac{\ell(\ell+1)}{r^2} f(r) \qquad \frac{dr_*}{dr} = \sqrt{h(r)/f(r)}$$

Regularity at the center: $V_{\ell=0}(0) = -\frac{1}{6} f(0) R(0)$

Scalar perturbations

We see the effect of the anomaly on the motion of scalar

$$V_{\ell=0}(0) = -\frac{1}{6} f(0) R(0) \quad \sim -\frac{m^2}{\sqrt{\rho}} M_P^{-2} \left(m^2 \sqrt{\rho} - M_P^{-4} \rho^2 \right)$$



The potential becomes less attractive

$$-\partial_{r_*}^2 u + V(r_*) u = \omega^2 u$$

$$\frac{m^2}{M_P} \sim 10^{-2} s^{-1}$$

Characteristic frequencies are measurable despite the very high energies involved

Conclusions

The effects of the trace anomaly become macroscopic at

$$\rho_c \sim (m/M_P)^{\frac{4(1-\alpha)}{2-\alpha}} \rho_P \ll \rho_P$$

$$\alpha < 1$$

$$m \ll M_P$$

and the curvature becomes negative $R < 0$

For scalar perturbations of stars close to Buchdahl limit, the anomaly controls the low frequency spectrum, and the object becomes 'less attractive'.

Outlook

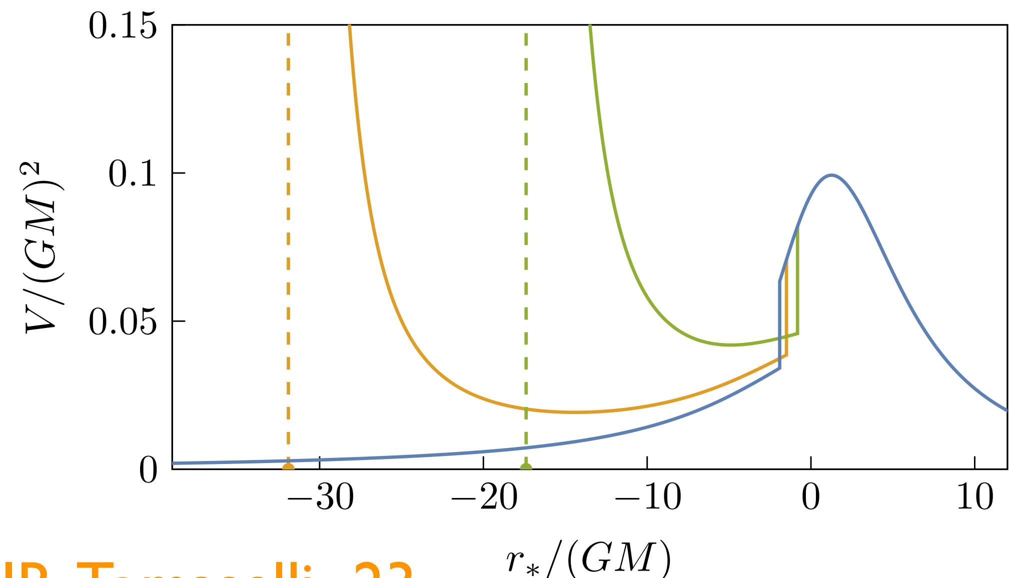
- Spin perturbations
- Gravitational collapse

Gravitational potential

Schwarzschild constant density sphere

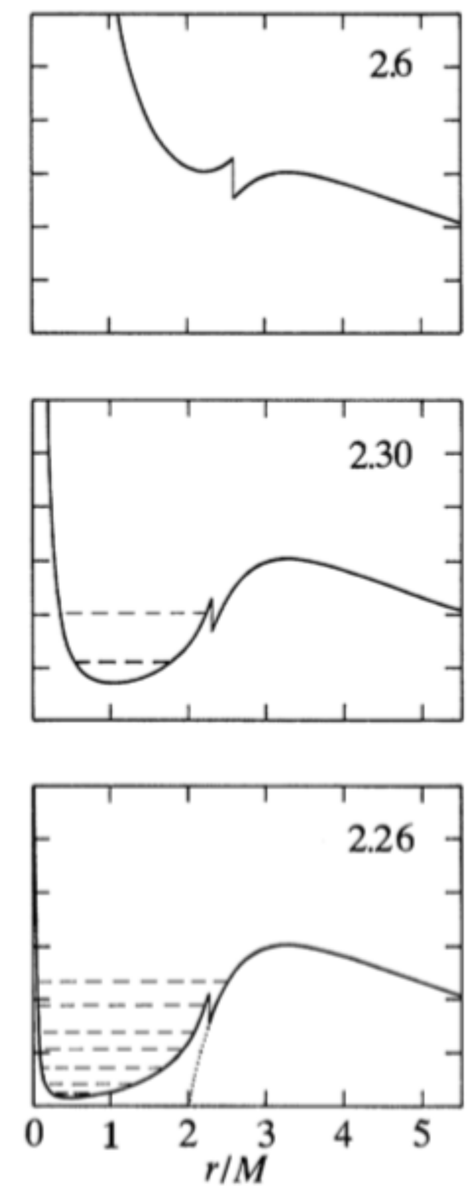
Spectrum of quasi-bound states in the Buchdahl limit

$V_\ell(r)$ spin 0



IR-Tomaselli -23

spin 2



Chandrasekhar-Ferrari ´92

In QFT the vacuum cannot be 'turned off'